Suggested Solution of Exercises for Quiz 1

Question 1. Suppose that $\lim a_n = 3$. Show that

$$\lim_{n\to\infty} \frac{a_n^2+1}{a_n-2} = 10.$$

Solution. Note that

$$\left|\frac{a_n^2+1}{a_n-2} - 10\right| = \left|\frac{a_n^2-10a_n+21}{a_n-2}\right| = \frac{|a_n-7|}{|a_n-2|}|a_n-3|.$$

Since $\lim a_n = 3$, there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - 3| < \frac{1}{2}, \quad \forall n \geq N_1.$$

i.e., $-0.5 < a_n - 3 < 0.5$. Hence for any $n \geq N_1$,

$$0.5 < |a_n - 2| < 1.5 \quad \text{and} \quad 3.5 < |a_n - 7| < 4.5.$$

Let $\varepsilon > 0$. There exists $N_2 \in \mathbb{N}$ such that

$$|a_n - 3| < \frac{\varepsilon}{9}, \quad \forall n \geq N_2.$$

Take $N = \max\{N_1, N_2\}$. Then for any $n \geq N$,

$$\left|\frac{a_n^2+1}{a_n-2} - 10\right| = \frac{|a_n-7|}{|a_n-2|}|a_n-3| < \frac{4.5}{0.5}\cdot\frac{\varepsilon}{9} = \varepsilon.$$

The result follows.

Question 2. Let (x_n) be a sequence of non-negative numbers. Suppose that

$$\lim_{n\to\infty} (-1)^n x_n$$

exists in $\mathbb{R}$. Show that (x_n) converges and find its limit.

Solution. Since $((-1)^n x_n)$ is convergent, its subsequences $((-1)^{2n}x_{2n})$ and $((-1)^{2n-1}x_{2n-1})$ are both convergent and have the same limit. Note that for each n,

$$(-1)^{2n}x_{2n} = x_{2n} \geq 0 \quad \text{and} \quad (-1)^{2n-1}x_{2n-1} = -x_{2n-1} \leq 0.$$

It follows that

$$0 \leq \lim((-1)^{2n}x_{2n}) = \lim((-1)^n x_n) = \lim(-1)^{2n-1}x_{2n-1} \leq 0.$$

Hence $\lim\limits_{n\to\infty}(-1)^n x_n = 0$. Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}|x_n| = \lim_{n\to\infty}|(-1)^n x_n| = \left|\lim_{n\to\infty}(-1)^n x_n\right| = 0.$$

Question 3. Let (x_n) be a bounded sequence of real numbers. Define

$$E = \{x \in \mathbb{R} : \text{there is a subsequence } (x_{n_k}) \text{ of } (x_n) \text{ that converges to } x.\}$$

Let $\alpha = \overline{\lim}\, x_n$. Show that $\alpha \in E$ and $\alpha = \sup E$.

Solution. Note that

$$\alpha = \overline{\lim}\, x_n = \lim_k \left(\sup_{n \geq k} x_n \right) = \lim u_k, \quad \text{where } u_k = \sup_{n \geq k} x_n.$$

To show that $\alpha \in E$, we need to find a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) converges to α. Take $n_1 = 1$. For each $k \in \mathbb{N}$, take $n_{k+1} > n_k$ such that

$$u_{n_k+1} - \frac{1}{k} < x_{n_k} \leq u_{n_k+1}.$$

By Squeeze theorem, it follows that

$$\alpha = \lim x_{n_k}.$$

To show that $\alpha = \sup E$, first note that $\alpha \leq \sup E$ because we have shown that $\alpha \in E$. On the other hand, let $x \in E$ and (x_{n_k}) be a subsequence of (x_n) that converges to x. Since $n_k \geq k$ for all $k \in \mathbb{N}$,

$$x_{n_k} \leq u_k, \quad \forall k \in \mathbb{N}.$$

Hence $x \leq \alpha$. Since $x \in E$ is arbitrary, $\sup E \leq \alpha$.

Question 4. Let a be a positive real number and $x_1 > \sqrt{a}$. Define the sequence (x_n) by

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right).$$

Prove that (x_n) is convergent and $\lim x_n = \sqrt{a}$.

Solution. We first show that $x_n \geq \sqrt{a}$ for all $n \in \mathbb{N}$ by induction. The case for $n = 1$ is given. Note that

$$x_{n+1}^2 = \frac{1}{4} \left(x_n + \frac{a}{x_n} \right)^2 = \frac{1}{4} \left(x_n - \frac{a}{x_n} \right)^2 + a \geq a.$$

Since obviously $x_n \geq 0$ for all $n \in \mathbb{N}$, it follows that $x_{n+1} \geq \sqrt{a}$. Now, for any $n \in \mathbb{N}$,

$$x_{n+1} - x_n = \frac{1}{2} \left(\frac{a}{x_n} - x_n \right) \leq \frac{1}{2} \left(\frac{a}{\sqrt{a}} - \sqrt{a} \right) = 0.$$

Hence (x_n) is a decreasing sequence and it is bounded below by $\sqrt{a}$. By Monotone Convergence Theorem, (x_n) is convergent. Let $x = \lim x_n$. Then

$$x = \frac{1}{2} \left(x + \frac{a}{x} \right).$$

Solving gives $x = \pm\sqrt{a}$. Since $x_n \geq \sqrt{a}$ for all $n \in \mathbb{N}$, $x \geq \sqrt{a}$. It follows that $x = \sqrt{a}$.

Question 5. Prove the **Nested Interval Property** by using **Bolzano-Weierstrass Theorem**.

Solution. Let $I_n = [a_n, b_n]$ be a nested sequence of closed bounded intervals. We need to find a $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$ by Bolzano-Weierstrass Theorem.

Consider the sequence (a_n) of real numbers. By Bolzano-Weierstrass Theorem, there exists a subsequence (a_{n_k}) of (a_n) that converges to some $\xi \in \mathbb{R}$. We need to show that $\xi \in I_n$ for all $n \in \mathbb{N}$. Note that for each $n \in \mathbb{N}$, there exists $K \in \mathbb{N}$ such that $n_K \geq n$. Hence

$$a_n \leq a_{n_K} \leq a_{n_k} \leq b_{n_K} \leq b_n, \quad \forall k \geq K.$$

In particular,

$$a_n \leq a_{n_k} \leq b_n, \quad \forall k \geq K.$$

Taking limit as $k \to \infty$, it follows that

$$a_n \leq \xi \leq b_n.$$

i.e., $\xi \in I_n$.